

Effective Lagrangian of Domain Wall Networks

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Abstract. Domain wall networks are studied in $\mathcal{N} = 2$ supersymmetric $U(N_C)$ gauge theory with $N_F(> N_C)$ flavors. We find a systematic method to construct domain wall networks in terms of moduli matrices. Normalizable moduli parameters of the network are found to be sizes and phases of the loop. We obtain moduli space metric which specifies the effective Lagrangian on the domain wall networks. It is used to study dynamics of domain wall networks with the moduli approximation.

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1 Introduction

D-branes play an essential role in understanding non-perturbative dynamics in string theories. As an extended object preserving half of the supersymmetry (SUSY), they have many similarities with domain walls in field theories. More than two domain walls intersect or meet with angles in general, and networks or webs of these objects emerge when many domain walls meet at various junctions. Junctions and networks of domain walls have many similarities with those of D-branes. Both of them can preserve a quarter of SUSY. Therefore they are called 1/4 BPS states. A few exact solutions of domain wall junctions have been obtained some time ago [1].

The purpose of the present paper is to give a compact summary of our study of domain wall networks in $U(N_C)$ gauge theories with $N_F(> N_C)$ Higgs scalars in the fundamental representations. 1/2 BPS parallel walls can exist for hypermultiplets with real masses in supersymmetric theories with eight supercharges in spacetime dimensions $d \leq 5$ [2,3]. On the other hand, to obtain non-parallel walls, we need complex masses for hypermultiplets which is possible in $d \leq 4$. We find that genuine moduli corresponding to normalizable modes are sizes of wall loops and their associated phases. We obtain the metric of these moduli in the effective Lagrangian of domain wall networks. The effective Lagrangian allows us to discuss dynamics of domain wall networks in the moduli approximation.

More detailed analysis of some topics on domain wall networks may be found in our recent papers [4], and a general survey of the moduli matrix approach is given in our review [5]. As a review for other solitons in the same model, see [6].

2 1/4 BPS Equations

2.1 SUSY $U(N_C)$ Gauge Theory with N_F Flavors

We consider 3+1 dimensional $\mathcal{N} = 2$ supersymmetric $U(N_C)$ gauge theory with $N_F(> N_C)$ massive hypermultiplets in the fundamental representation. The physical bosonic fields contained in this model are a gauge field W_μ ($\mu = 0, 1, 2, 3$) and real adjoint scalars Σ_α ($\alpha = 1, 2$) in the vector multiplet, and N_F complex doublets of scalars H^{iAr} ($r = 1, 2, \dots, N_C$, $A = 1, 2, \dots, N_F$, $i = 1, 2$) in the hypermultiplets. We express $N_C \times N_F$ matrix of the hypermultiplets by H^i . With the metric $\eta_{\mu\nu} = (+1, -1, -1, -1)$, we obtain the bosonic Lagrangian with the gauge coupling g

$$\mathcal{L} = \text{Tr} \left[-\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{g^2} \sum_{\alpha=1}^2 \mathcal{D}_\mu \Sigma_\alpha \mathcal{D}^\mu \Sigma_\alpha + \mathcal{D}_\mu H^i (\mathcal{D}^\mu H^i)^\dagger \right] - V, \quad (1)$$

$$V = \text{Tr} \left[\sum_{\alpha=1}^2 (H^i M_\alpha - \Sigma_\alpha H^i) (H^i M_\alpha - \Sigma_\alpha H^i)^\dagger - \frac{1}{g^2} [\Sigma_1, \Sigma_2]^2 + \frac{g^2}{4} (H^1 H^{1\dagger} - H^2 H^{2\dagger} - c \mathbf{1}_{N_C})^2 \right], \quad (2)$$

with diagonal mass matrices $M_1 = \text{diag}(m_1, m_2, \dots, m_{N_F})$ and $M_2 = \text{diag}(n_1, n_2, \dots, n_{N_F})$, and $c > 0$ the Fayet-Iliopoulos (FI) parameter. The covariant derivatives

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and the field strength are defined by $\mathcal{D}_\mu \Sigma_\alpha = \partial_\mu \Sigma_\alpha + i[W_\mu, \Sigma_\alpha]$, $\mathcal{D}_\mu H^i = \partial_\mu H^i + iW_\mu H^i$ and $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + i[W_\mu, W_\nu]$, respectively.

2.2 Vacua and BPS Equations

Supersymmetric vacuum is characterized by a set of N_C different flavor indices $\langle A_1 A_2 \cdots A_{N_C} \rangle$ out of N_F flavors, which correspond to the color component of nonvanishing hypermultiplet [8]

$$H^{1rA} = \sqrt{c} \delta^{Ar} A, \quad H^{2rA} = 0, \quad (3)$$

$$\Sigma \equiv \Sigma_1 + i\Sigma_2 \quad (4)$$

$$= \text{diag} \left(m_{A_1} + in_{A_1}, \dots, m_{A_{N_C}} + in_{A_{N_C}} \right).$$

This is the Higgs phase, where only the domain walls and vortices can exist as elementary solitons. Monopoles and instantons can exist when they accompany vortices to form composite solitons [6, 5].

By assuming dependence on x^1, x^2 and requiring 1/4 of supercharges to be conserved, we find the 1/4 BPS equations for the web of walls

$$[\mathcal{D}_1 + \Sigma_1, \mathcal{D}_2 + \Sigma_2] = 0, \quad \mathcal{D}_\alpha H = H M_\alpha - \Sigma_\alpha H, \quad (5)$$

$$\sum_\alpha \mathcal{D}_\alpha \Sigma_\alpha = \frac{g^2}{2} (c \mathbf{1}_{N_C} - H H^\dagger), \quad (6)$$

with $\alpha = 1, 2$. Solutions of the 1/4 BPS equations saturate the lower bound of the energy E of the field configurations, which is given by the sum of topological charges (α in Z_α is not summed)

$$E \geq Z_1 + Z_2 + Y, \quad Z_\alpha \equiv \int d^2x c \partial_\alpha \text{Tr} \Sigma_\alpha, \quad (7)$$

$$Y \equiv \int d^2x \frac{2}{g^2} \partial_\alpha \text{Tr} (\epsilon^{\alpha\beta} \Sigma_2 \mathcal{D}_\beta \Sigma_1)$$

2.3 BPS Solutions and Moduli Space

The first equation in Eqs.(5) is the integrability condition for the second equation in Eqs.(5), whose solutions are obtained in terms of $N_C \times N_C$ non-singular matrix $S(x^\alpha)$ as

$$H = S^{-1} H_0 e^{M_1 x^1 + M_2 x^2}, \quad W_1 - i\Sigma_1 = -iS^{-1} \partial_1 S, \quad (8)$$

$$W_2 - i\Sigma_2 = -iS^{-1} \partial_2 S.$$

Here H_0 is an $N_C \times N_F$ constant complex matrix of rank N_C . We call H_0 the *moduli matrix* because it contains moduli parameters of solutions as we see below. Defining a gauge invariant matrix $\Omega \equiv S S^\dagger$, Eq.(6) can be written as

$$\frac{1}{cg^2} [\partial_\alpha (\partial_\alpha \Omega \Omega^{-1})] = \mathbf{1}_{N_C} - \Omega_0 \Omega^{-1}, \quad (9)$$

with $\Omega_0 \equiv c^{-1} H_0 e^{2(M_1 x^1 + M_2 x^2)} H_0^\dagger$. We call Eq. (9) the *master equation*, and assume the existence and the uniqueness of the solution once Ω_0 is given.

Since the same physical configurations are realized by (H_0, S) and (H'_0, S') related by the *V-symmetry* with $V \in GL(N_C, \mathbf{C})$

$$H_0 \rightarrow H'_0 = V H_0, \quad S \rightarrow S' = V S, \quad (10)$$

we obtain the *total* moduli space of the web of domain walls as the complex Grassmann manifold

$$G_{N_F, N_C} \simeq \{H_0 \mid H_0 \sim V H_0, V \in GL(N_C, \mathbf{C})\} \\ \simeq SU(N_F) / [SU(N_F - N_C) \times SU(N_C) \times U(1)]. \quad (11)$$

This space is made by gluing all the topological sectors together and is not endowed with a metric [3].

In strong gauge coupling limit $g^2 \rightarrow \infty$, Eq. (9) can be algebraically solved. For instance, in the case of Abelian gauge theory ($N_C = 1$) the strong coupling limit gives configurations of scalar fields up to gauge symmetry as

$$H^A = \sqrt{c} \frac{H_0^A e^{m_A x^1 + n_A x^2}}{\sqrt{\sum_{B=1}^{N_F} |H_0^B|^2 e^{2(m_B x^1 + n_B x^2)}}}. \quad (12)$$

3 Webs of Walls

The moduli matrix of $U(1)$ gauge theory is given by

$$H_0 = \sqrt{c} (e^{a_1 + ib_1}, \dots, e^{a_{N_F} + ib_{N_F}}). \quad (13)$$

In the case of $N_F = 2$, two vacua exist and a wall connecting them is located where two vacuum weights are equal

$$(m_1 - m_2)x^1 + (n_1 - n_2)x^2 + a_1 - a_2 = 0. \quad (14)$$

In the case of $N_F = 3$, there exist 3 discrete vacua labeled by $\langle A \rangle$ ($A = 1, 2, 3$), and three walls can meet to form a junction as illustrated in Fig.1. We can re-

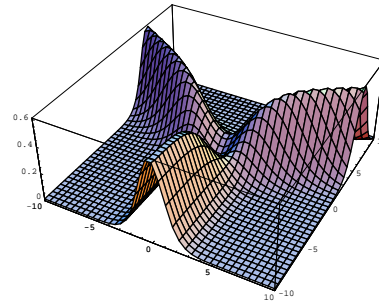


Fig. 1. Binding energy at the junction point: The energy density is numerically evaluated for the moduli matrix $H_0 e^{m \cdot x} = (e^{x^2}, e^{\sqrt{3}x^1/2 - x^2/2}, e^{-\sqrt{3}x^1/2 - x^2/2})$, gauge coupling $g = 1$ and FI parameter $c = 1$.

ognize that the topological charge Y associated to the junction gives a negative contribution to the energy density, which can be interpreted as binding energy of

domain walls. This is a feature in the $U(1)$ gauge theory. The $1/4$ BPS wall junction is characterized by a triangle with 3 vertices at $m_A + in_A$ in Σ plane. We call such polygons in the Σ plane as grid diagrams.

The models with $N_F \geq 4$ admit more ample webs of walls. Let us take the case of $N_F = 4$ model. In a choice of mass parameters, we obtain tree diagram as shown in Fig.2. In another choice of mass parameters, we obtain a loop diagram as shown in Fig.3.

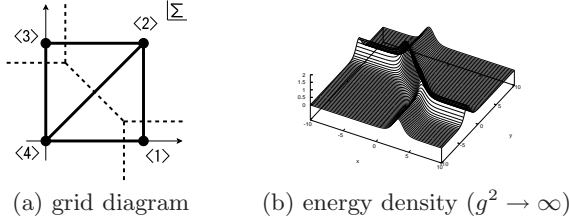


Fig. 2. Wall web with 4 external legs of walls. Grid diagram:(a), and energy density:(b). ($[m_A, n_A] = \{[1, 0], [1, 1], [0, 1], [0, 0]\}$)

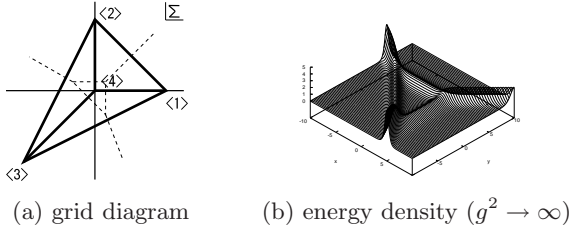


Fig. 3. Web with 1 loop in the $N_F = 4$ model. Grid diagram:(a), and energy density:(b). ($[m_A, n_A] = [1, 0], [0, 1], [-1, -1], [0, 0]$)

In the case of non-Abelian gauge theories, we can obtain positive contributions to the topological charge Y . We find in that case that the junction contains the Hitchin system which arises as a dimensionally reduced Yang-Mills instanton.

4 Effective Lagrangian of Wall Loops

To obtain effective Lagrangian on the webs of walls, we need to fix boundary conditions at asymptotic infinity. With that restriction, we can only fluctuate moduli parameters associated with the loop of the web. Therefore the sizes of the loops and their associated phases are the only normalizable moduli which can be promoted to fields on the web of walls. We find a general formula for the effective Lagrangian [7]

$$\mathcal{L}^{eff} = K_{ij^*}(\phi, \phi^*) \partial^\mu \phi^i \partial_\mu \phi^{j^*},$$

$$K(\phi, \phi^*) = K_w(\phi, \phi^*) + K_g(\phi, \phi^*) \quad (15)$$

$$K_w(\phi, \phi^*) \equiv \int d^2x c \log \det \Omega, \quad (16)$$

$$K_g(\phi, \phi^*) \equiv \int d^2x \frac{1}{2g^2} \text{Tr}(\Omega^{-1} \partial_\alpha \Omega)^2. \quad (17)$$

Let us take as an example, $N_F = 4$ case in $U(1)$ gauge theory with the mass assignment $[m_4, n_4] = [0, 0]$ where we choose the moduli matrix with the complex moduli $\phi = e^{r+i\theta}$

$$H_0 = \sqrt{c}(1, 1, \phi). \quad (18)$$

4.1 Metric at Small Loops

We can explicitly evaluate the metric at small loops in the strong coupling limit $g^2 \rightarrow \infty$

$$\begin{aligned} K_w &\equiv c \int d^2x \left[\log \Omega_0 - \log \tilde{\Omega}_0 \right] \\ &= c \int d^2x \log \left(1 + \frac{|\phi|^2}{\tilde{\Omega}_0} \right), \end{aligned} \quad (19)$$

$$\Omega_0 = e^{2\mathbf{m}_1 \cdot \mathbf{x}} + e^{2\mathbf{m}_2 \cdot \mathbf{x}} + e^{2\mathbf{m}_3 \cdot \mathbf{x}} + |\phi|^2, \quad (20)$$

$$\tilde{\Omega}_0 = e^{2\mathbf{m}_1 \cdot \mathbf{x}} + e^{2\mathbf{m}_2 \cdot \mathbf{x}} + e^{2\mathbf{m}_3 \cdot \mathbf{x}}, \quad (21)$$

where the mass of the hypermultiplet of the A -th flavor is denoted by a two-vector $\mathbf{m}_A = (m_A, n_A)$, $\mathbf{m}_A \cdot \mathbf{x} \equiv m_A x^1 + n_A x^2$, and $\Delta_{[123]}$ is the area of the triangle in field space with the masses of 1, 2, 3 as vertices.

Let us define $\alpha_i \equiv (\mathbf{m}_j \times \mathbf{m}_k) / \Delta_{[123]}$. We find that we can expand the integrand in powers of $|\phi|$ in the region of $|\phi|^2 \leq \exp(-\sum \alpha_i \log \alpha_i)$. Therefore we know that there exists well-defined smooth function even for $|\phi|^2 \geq \exp(-\sum \alpha_i \log \alpha_i)$. Moreover, this can be written as a sum of hypergeometric functions in cases where α_i are rational numbers. We obtain the Kähler

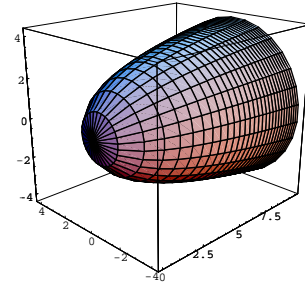


Fig. 4. The moduli space of single triangle loop around $\phi = 0$ where the loop shrinks. $U(1)$ isometry is the phase modulus. The other direction is the size modulus of the loop.

potential for $|\phi|^2 \leq \exp(-\sum \alpha_i \log \alpha_i)$

$$K_w = \frac{c}{4\Delta_{[123]}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{\Gamma(\alpha_1 n) \Gamma(\alpha_2 n) \Gamma(\alpha_3 n)}{\Gamma(n)} |\phi|^{2n}.$$

We find that the scalar curvature is finite (nonsingular) even at $\phi = 0$ (vanishing loop)

$$R = \frac{16\Delta_{[123]}}{c} \frac{\Gamma(2\alpha_1) \Gamma(2\alpha_2) \Gamma(2\alpha_3)}{(\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\alpha_3))^2} + \mathcal{O}(|\phi|^2). \quad (22)$$

4.2 Metric at Large Loops

For large sizes of the loop, Ω in the integrand of the Kähler potential (16) and (17) is dominated by the vacuum weight in each region, apart from finite regions near walls and junctions. Therefore evaluation in terms of vacuum weights is a good approximation at large values of $|\phi|$. This is the so-called tropical limit, and the approximation is valid for arbitrary values of gauge coupling g . By using the tropical approximation, we find K_w is given by c times the volume of the tetrahedron

$$K_w^{trop} = \frac{c}{24\Delta_{[123]}} \frac{1}{\alpha_1\alpha_2\alpha_3} (\log |\phi|^2)^3, \quad (23)$$

$$K_g^{trop} = -\frac{2}{g^2} |\mathbf{m}_1|^2 A_1 - \frac{2}{g^2} |\mathbf{m}_2|^2 A_2 - \frac{2}{g^2} |\mathbf{m}_3|^2 A_3 \\ = -\frac{(\log |\phi|^2)^2}{4g^2\Delta_{[123]}} \left(\frac{|\mathbf{m}_{12}|^2}{\alpha_3} + \frac{|\mathbf{m}_{23}|^2}{\alpha_1} + \frac{|\mathbf{m}_{31}|^2}{\alpha_2} \right). \quad (24)$$

Combining the above Kähler potentials (23) and (24), we obtain the total Kähler metric as

$$ds^2 = \frac{c}{\Delta_{[123]}} \left[\frac{r}{\alpha_1\alpha_2\alpha_3} \right. \\ \left. - \frac{1}{g^2c} \left(\frac{|\mathbf{m}_{12}|^2}{\alpha_3} + \frac{|\mathbf{m}_{23}|^2}{\alpha_1} + \frac{|\mathbf{m}_{31}|^2}{\alpha_2} \right) \right] (m^2 dr^2 + d\theta^2). \quad (25)$$

We can understand the effective action as the kinetic energies of walls and junctions due to the moduli motion.

5 Dynamics of Loops

Let us finally comment on the dynamics of domain wall loops by using the effective Lagrangian with the moduli approximation. The fan-like metric in Fig.4 implies that the domain wall loops tend to expand without limit. Non-Abelian gauge theory allows different types of loops which can be deformed to each other by changing a modulus. In this case, the moduli geometry looks like a sandglass made by gluing the tips of the two cigar-(cone)-like metrics of a single triangle loop.

Then the sizes of all loops tend to grow for a late time in general models with complex Higgs masses, while the sizes are stabilized at some values once triplet masses are introduced for the Higgs fields.

Dynamics of a double loop in Abelian gauge theory and a non-Abelian loop is discussed in the last paper of [4].

6 Conclusion

1. Webs of domain walls are constructed as $1/4$ BPS states in $\mathcal{N} = 2$ SUSY $U(N_C)$ non-Abelian gauge theories in 4 dimensions with $N_F(> N_C)$ hypermultiplets in the fundamental representation.

2. Total moduli space of the webs of walls is given by a complex Grassmann manifold G_{N_F, N_C} described by the moduli matrix H_0 .
3. Exact solutions of webs of walls are obtained for $g^2 \rightarrow \infty$.
4. Abelian junction has a topological charge contributing negatively to the energy (binding energy). Non-Abelian junction has a topological charge contributing positively to energy. The positive charge is the result of the Hitchin system sitting at the junction.
5. A general formula for the effective Lagrangian is obtained.
6. Normalizable moduli of web of walls are sizes of the loop and their associated phases.
7. Metric of a single triangle loop of walls is explicitly worked out and the dynamics of loops are worked out.

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